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Multibody approach for tolerance analysis and optimization of mechanical systems

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Abstract

A multibody approach is suitable for tolerance analysis of mechanical systems since multibody formulation can directly consider part-level tolerance variables. In this study, procedures for performing tolerance analysis and corresponding sensitivity analysis for spatial multibody systems are proposed. First, statistical formulation for performing multibody system tolerance analysis is developed to obtain system level tolerance for given part-level tolerances. One very useful aspect of the proposed formulation is that in the process of computing system tolerance, the sensitivity of system tolerance with respect to part-level tolerances can be additionally obtained. The kinematics of spatial multibody systems has been redefined in terms of both generalized coordinates and part-level tolerance variables. Tolerances in geometry of a body are specified in terms of the variations in relative locations of joint definition points and relative distance between them. Tolerances in the joint kinematics are defined through variations in vector closure equations and orthogonality equations that are two fundamental constraint equations for most kinematic joints. To demonstrate the validity and effectiveness of the proposed tolerance analysis procedure, tolerance analysis of a spatial 4-bar mechanism and tolerance optimization are performed.

Keywords: Multibody system; Tolerance analysis; Kinematic analysis; Sensitivity analysis; Optimization

1. Introduction

A multibody approach for analysis of mechanical systems has a major advantage in that it can account for part-level design variables in the evaluation of the overall performance of the system. Meanwhile, tolerance analysis determines the effect of tolerances of individual parts on the assembly system. Thus, the multibody approach is inherently suitable for tolerance analysis of mechanical systems. As parts are assembled, the tolerances of each part accumulate to form a tolerance stack-up. Therefore, many small tolerance variations can add together to form a large residual stack-up, which can affect system performance and cost. Even though tolerance assignment forms an important link between design and manufacturing processes, designers often neglect tolerance or view it as a trivial part of design. To overcome this kind of thinking, engineers must be provided with tools that will allow them to understand the consequences of tolerance assignment and the relationship between tolerance and product performance.

The kinematics of multibody systems can be defined in terms of generalized coordinates and design variables. While generalized coordinates determine positions and velocities as a function of time, design variables specify the kinematic configuration of the system, such as link length, joint location, direction of joint motion axis, etc. To carry out a tolerance analysis of a multibody system, multibody system kinematics should take into consideration the tolerance vari-

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ables of each part, and then a systematic process to perform kinematic analysis should be established to obtain the assembled system-level tolerance. Furthermore, if the sensitivity of the system tolerance with respect to part-level tolerance variables can be obtained, mechanical tolerance can be optimized.

There have been many efforts to analyze kinematic tolerances of mechanical systems. The equivalent linkage model by Dhande[1] and Chakraborty[2] is an approximate method that adopts imaginary linkages to account for the combined errors in link length and joint clearance. The method is applied to the tolerance analysis of a planar 4-bar path generator, which has link length errors and joint clearances. Mallik and Dhande[3] used the equivalent linkage model in the analysis of planar path generation. Rhyu and Kwak[4] proposed a modified form of the equivalent linkage model for analysis and control of tolerances in a planar 4-bar mechanism. The equivalent linkage model, however, has some drawbacks in terms of design optimization. First, new equivalent linkage must be created whenever kinematic configurations are changed, and second, the effects of link length tolerance and joint clearances cannot be separately examined. The effective link model proposed by Lee [5-7] is an improved version of the equivalent linkage model, in that the effective link model does not require a new modeling process even if joint configurations are changed. However, the effective link model also has a limitation with respect to the accuracy of analysis since the effective link cannot account for the change of the joint motion axis angle. To examine the link length tolerance and the joint clearance independently, Choi et al.[8] proposed the clearance vector model, which can be applied to revolute joints of planar mechanisms. By defining the clearance vector from the nominal center of a joint to the actual center of the joint, the joint tolerance can be separated from the tolerance of link length. This method is applied to the tolerance analysis and design of a planar 4-bar path generator with lubricated joints. Choi et al.[9] also demonstrated that the clearance vector model is more accurate than the equivalent linkage model after comparing the analysis results of planar mechanisms.

Stoenescu and Marghitu [10] investigated the dynamic behavior of a planar, rigid-link mechanism with a sliding joint clearance. They considered the clearance of link length. When a topology change occurs, the equations of motion are reformulated to reflect the changes of system topology. More recently, Choi[11] presented a general framework and computational algorithms for statistical tolerance analysis and modal analysis of multibody systems based on general multibody formulations. His work is applicable to a broad range of multibody system analyses, including kinematic and dynamic analysis, sensitivity analysis, static and dynamic equilibrium analysis, modal analysis and tolerance analysis, and it takes a very similar approach to this work in performing tolerance analysis and sensitivity analysis. However, in this paper, emphasis is given to kinematic tolerance analysis taking more elaborate consideration to specify kinematic design variables and investigate their effects on the variations in joint geometry.

Previous works on tolerance analysis have been mostly limited to planar systems. The kinematics of planar and spatial systems is quite different in terms of geometric complexities associated with joint definitions. In planar systems, most of the lower pair joints can be modeled by using revolute and translational joints, whose constraint equations are relatively simple; however, for spatial systems, there are various types of lower pair joints such as spherical, universal, cylindrical, and screw joints, which have all a different form of kinematic characteristics and constraint equations. Moreover, the tolerance analysis of a spatial system is more complex and difficult because dimensional or geometric tolerances of a part, such as variations in flatness or cylindricity, are further amplified because of joint connections between adjoining parts. This study proposes a systematic procedure for tolerance analysis of spatial multibody systems, and in addition, an analytical method to perform sensitivity analysis and optimization of multibody system tolerance is developed.

2. Formulation for multibody system tolerance analysis

The kinematic behavior of a multibody system can be represented by m-constraint equations, which are functions of n-generalized coordinates $\mathbf{q} = [q_1, q_2, \cdots, q_n]^T$, and k-design variables $\mathbf{b} = [b_1, b_2, \cdots, b_k]^T$

$$\mathbf{\Phi}(\mathbf{q}(\mathbf{b},t), \mathbf{b}, t) = [\Phi_1, \Phi_2, \ldots, \Phi_m]^T \tag{1}$$

Design variables $\mathbf{b}$ that define the kinematic configuration of a multibody system, such as the length of link, direction of motion axis, and location of pivot, etc., are independent variables. Generalized coordinates $\mathbf{q} = \mathbf{q}(\mathbf{b}, t)$, which are implicit functions of the design variables and time, are dependent variables due to the kinematic constraint Eq. (1). According to the implicit function theorem [12], if the Jacobian of $\mathbf{\Phi}$ has full rank, generalized coordinates $\mathbf{q}$ can be solved from Eq. (1). If there exist tolerances in the constituting bodies and joints of a multibody system, design variables $\mathbf{b}$ are no longer constant, and should be considered as statistically distributed variables to account for the variations of kinematic configuration. Thus $\mathbf{b}$ can be represented by the mean $\boldsymbol{\mu}$ and tolerance $\mathbf{T}$ as

$$\mathbf{b} = \boldsymbol{\mu} + \mathbf{T} \tag{2}$$

Since it is known that the tolerances of mechanical systems can be generally represented by a normal distribution (Choi et al[13], Park et al.[14]), design variables $\mathbf{b}$ can be assumed to have a normal distribution. However, other types of distributions, such as Poisson or exponential distributions, can be employed if necessary.

Let's assume that it is desired that variables whose mean and variance are $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}^2$, respectively, are within $\boldsymbol{\mu} \pm 3\boldsymbol{\sigma}$. As shown in Fig.1, the probability that the variables are within this range is 99.7% in case of normal distribution. The relation between tolerance T_{b_i} and variance σ_{b_i} of the i-th design variable b_i can be given as [15]

$$\sigma_{b_i}^2 = T_{b_i}^2 \ / \ 9 \tag{3}$$

Tolerances of design variables $\mathbf{b}$ influence the kinematic behavior of a multibody system or the system tolerance. Let's assume that kinematic performance

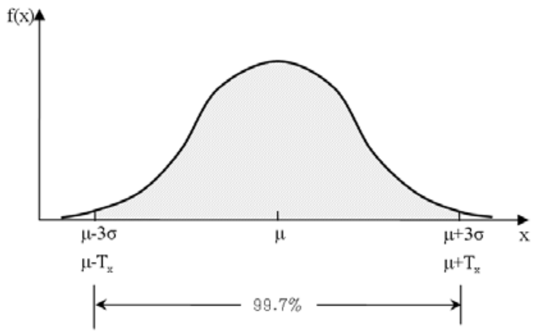


Fig. 1. Probability associated with normal distribution.

of a multibody system expressed in terms of generalized coordinates $\mathbf{q}$ and design variables $\mathbf{b}$ is represented by a variable Y.

$$\mathrm{Y} = \mathrm{Y}\,(\mathbf{q}(\mathbf{b}, \mathrm{t}), \mathbf{b}) \tag{4}$$

To obtain the mean value and variance of Y, Y is linearized about the mean values $\boldsymbol{\mu}$ of design variables $\mathbf{b}$ by using Taylor expansion as

$$\mathrm{Y} \approx \mathrm{Y}(\mathbf{q}, \mu_1, \ldots, \mu_k) + \sum_{i=1}^{k} (b_i - \mu_i) \left. \frac{d\mathrm{Y}}{db_i} \right|_{\mu_i} \tag{5}$$

Using the linearized equation, μ_{Y}, which is the mean value of Y, and σ_{Y}^2, which is the variance of Y, can be, respectively, given as

$$\mu_{\mathrm{Y}} = \mathrm{Y}(\mathbf{q}, \mu_1, \ldots, \mu_k) \tag{6}$$

$$\sigma_{\mathrm{Y}}^2 = \sum_{i=1}^{k} \sigma_{b_i}^2 \left(\left. \frac{d\mathrm{Y}}{db_i} \right|_{\mu_i} \right)^2 \tag{7}$$

where $d\mathrm{Y}/db_i$ represents the sensitivity of Y with respect to design variable b_i, and $\sigma_{b_i}^2$ is the variance of variable b_i. Once σ_{Y}^2 is known, the tolerance of Y can be obtained from Eq. (3). $d\mathrm{Y}/db_i$ can be calculated using the chain rule.

$$\frac{d\mathrm{Y}(\mathbf{q})}{db_i} = \sum_{p=1}^{n} \frac{\partial \mathrm{Y}}{\partial q_p} \frac{dq_p}{db_i} + \frac{\partial \mathrm{Y}}{\partial b_i} \tag{8}$$

Given $\mathrm{Y} = \mathrm{Y}\,(\mathbf{q}(\mathbf{b}, \mathrm{t}), \mathbf{b})$, then $\partial \mathrm{Y}/\partial q_p$ and $\partial \mathrm{Y}/\partial b_i$ can be directly computed; however, dq_p / db_i, which is the derivative of a generalized coordinate q_p with respect to b_i, cannot be directly calculated. To compute dq_p / db_i, Eq. (1) is differentiated with respect to b_i

$$\frac{d\Phi}{db_i} = \sum_{p=1}^{n} \frac{\partial \Phi}{\partial q_p} \frac{\partial q_p}{\partial b_i} + \frac{\partial \Phi}{\partial b_i} = 0 \tag{9}$$

Rearranging Eq. (9), the following equations are obtained.

$$\sum_{p=1}^{n} \frac{\partial \Phi}{\partial q_p} \frac{\partial q_p}{\partial b_i} = -\frac{\partial \Phi}{\partial b_i} \tag{10}$$

If kinematic constraint equations $\mathbf{\Phi}(\mathbf{q}(\mathbf{b}, \mathrm{t}), \mathbf{b}, \mathrm{t})$ are given in terms of generalized coordinates and design variables, then $\partial \Phi/\partial q_p$ and $\partial \Phi/\partial b_i$ can be computed. Thus dq_p / db_i can be obtained from Eq.

(10). The fact that $\partial Y/\partial b_i$, which is the sensitivity of function Y with respect to a design variable b_i, can be calculated means if Y is a cost function, then sensitivity analysis and optimization of tolerance can be performed.

3. Tolerance modeling for multibody system

Multibody systems can be regarded as an interconnection of bodies by various types of kinematic joints such as spherical, revolute, cylindrical, universal joints, etc. In a conventional kinematic analysis, constraint equations that represent kinematic joints are defined in terms of generalized coordinates only. However, in order to perform tolerance analysis, constraint equations should be expressed in terms of both generalized coordinates and tolerance variables. To begin with, let's consider the variations of the geometry of a body. Body i of a multibody system is shown in Fig. 2. The coordinate system x_i'-y_i'-z_i' represents the body reference frame with respect to inertial frame X-Y-Z. Vector $\mathbf{s}_i^P$ is the position vector to the origin of the x_i'-y_i'-z_i' frame, and points P and Q are the joint definition points through which body i is connected to other bodies by kinematic joints. Vectors $\mathbf{s}_i^P$ and $\mathbf{s}_i^Q$, respectively, define position vectors of points P and Q, and vector $\mathbf{d}$ is the position vector from point P to Q, which can be expressed as

$$\mathbf{d} = \mathbf{s}_i^Q - \mathbf{s}_i^P = \mathbf{A}_i\mathbf{s}_i'^Q - \mathbf{A}_i\mathbf{s}_i'^P \tag{11}$$

where $\mathbf{A}_i$ is the transformation matrix from the x_i'-y_i'-z_i' frame to the inertial frame, and δl and $\mathbf{s}_i'^Q$ are the local vector representations of $\mathbf{s}_i^P$ and $\mathbf{s}_i^Q$ in the x_i'-y_i'-z_i' frame, respectively.

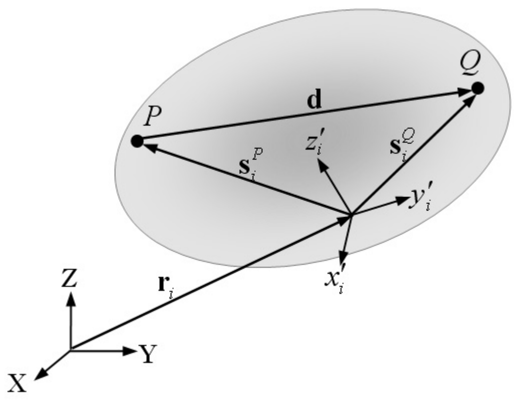


Fig. 2. Tolerance model for body geometry.

Both δl and $\mathbf{s}_i'^Q$ are time-constant vectors that define the joint definition points, which in fact determine the kinematic configuration of a body. Thus, variations in δl and $\mathbf{s}_i'^Q$ can account for the tolerances of the geometry of a body. In some cases, the relative position between P and Q determines the geometry of a body, and in other cases only the distance between P and Q is considered depending on the type of joints at P and Q. For example, if spherical joints are at both P and Q, then only the distance between P and Q is kinematically meaningful; however, when a revolute joint is at P and a translational joint is at Q, then the relative position between P and Q affects the kinematic behavior of the system.

To deal with the relative position tolerance between P and Q, one of δl, $\mathbf{s}_i'^Q$ is fixed, and the other can be regarded as a random variable. Here let's assume that δl is fixed and $\mathbf{s}_i'^Q$ is the random variable. Since the tolerance between P and Q can be in any direction, each element of $\mathbf{s}_i'^Q = [s_{ix}'^Q, s_{iy}'^Q, s_{iz}'^Q]^T$ can be an independent random variable with mean and tolerances given as

$$\mathbf{s}_i'^Q = \boldsymbol{\mu}(\mathbf{s}_i'^Q) + \mathbf{T}(\mathbf{s}_i'^Q) \tag{12}$$

If the tolerance follows the normal distribution and each tolerance is to be within the $\pm 3\boldsymbol{\sigma}$ range, then the relationship between variance and tolerance can be obtained by Eq. (3).

Distance l between P and Q is computed as

$$l^2 = \mathbf{d}^T\mathbf{d} = (\mathbf{A}_i\mathbf{s}_i'^Q - \mathbf{A}_i\mathbf{s}_i'^P)^T(\mathbf{A}_i\mathbf{s}_i'^Q - \mathbf{A}_i\mathbf{s}_i'^P) \tag{13}$$

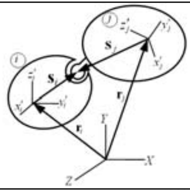
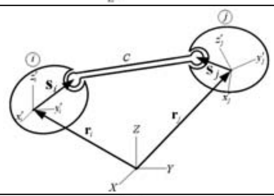
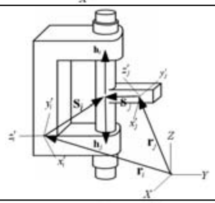
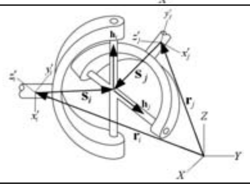
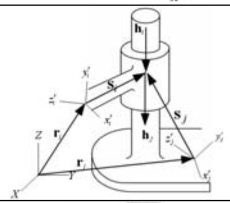
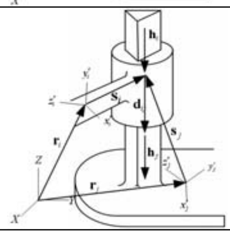
If it is assumed that δl is fixed and $\mathbf{s}_i'^Q$ is designated to be a random variable to account for the link length variation, then link length tolerance δl can be obtained by taking the variation of Eq. (13) as

$$l \cdot \delta l = (\mathbf{A}_i\mathbf{s}_i'^Q - \mathbf{A}_i\mathbf{s}_i'^P)^T(\mathbf{A}_i\,\delta\mathbf{s}_i'^Q - \mathbf{A}_i\mathbf{s}_i'^P) \tag{14}$$

In the above equation, variation $\delta\mathbf{s}_i'^Q$ is the same as the tolerance $\mathbf{T}(\mathbf{s}_i'^Q)$ in Eq. (12). Thus if the tolerance of $\delta\mathbf{s}_i'^Q$ is defined, then the tolerance in the link length can be calculated by Eq. (14).

In multibody system modeling, various kinds of kinematic joints are used. In Table 1, some of the most frequently used joints are listed with a brief explanation of each constraint equation. Among the constraint equations, there are two common types: vector

Table 1. Kinematic joints and their constraint equations.

Joint	Respresentation	Contraints [# of Equations]	d.o.f
Spherical		Vector closure [3] $\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i = 0$	3
Distance		Constant distance [1] $(\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i)^T (\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i) - C^2 = 0$	5
Revolute		Vector closure [3] $\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i = 0$ Orthogonality [2] $\mathbf{h}_i^T \mathbf{f}_j = 0$ * $\mathbf{h}_i^T \mathbf{g}_j = 0$ *	1
Universal		Vector closure [3] $\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i = 0$ Orthogonality [1] $\mathbf{h}_i^T \mathbf{h}_j = 0$ *	2
Cylindrical		Vector closure [3] $\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i = 0$ Orthogonality [1] $\mathbf{h}_i^T \mathbf{f}_j = 0$ *	2
Translational		Vector closure [3] $\mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i = 0$ Orthogonality [2] $\mathbf{h}_i^T \mathbf{f}_j = 0$ $\mathbf{h}_i^T \mathbf{d}_j = 0$ **	1

closure equations and orthogonality equations. Vector closure equations require that at a common point between two bodies, two bodies must maintain contact at all times. These equations are used in spherical, revolute, and universal joints. On the other hand, orthogonality equations impose a condition in which any two axes involved must always be perpendicular to each other. These equations are used to define the direction of the joint motion axis. For example, one orthogonality equation that enforces the cross shape of the universal joint spider, and one vector closer equation define a universal joint. Orthogonality equations are also used in revolute, cylindrical, and translational joints. In a revolute joint, two orthogonal constraints are used, while in cylindrical and translational joints, 4 and 5 orthogonal constraints are used, respectively. A distance constraint equation and its tolerance model are given in Eq. (14).

Spherical and revolute joints are examined in detail to explain the variations of joint kinematics because a spherical joint has closure equations, and a revolute joint has additional orthogonality equations as well as vector closure equations. Since the constraint equations for other types of joints are just combinations of vector closure equations and orthogonality equations, their tolerance models can be similarly derived referring to the tolerance model of a spherical and revolute joint.

A spherical joint composed of the housing and ball is shown in Fig. 2. It has clearance due to a gap and/or imperfect shape of the housing and ball. Vector $\mathbf{r}_i$ is the position vector from the inertial frame X-Y-Z to the origin of the local coordinate x_i'-y_i'-z_i' of housing ⓘ, and $\mathbf{r}_j$ is the position vector from the origin of the inertial X-Y-Z frame to the origin of the local frame x_j'-y_j'-z_j' of the ball ⓙ. Vector $\mathbf{s}_i'$ is the position vector from the origin of x_i'-y_i'-z_i' to the center P_i of the perfect circular housing, and $\mathbf{s}_j'$ is the position vector from the origin of x_j'-y_j'-z_j' to P_j. $\mathbf{A}_i$ and $\mathbf{A}_j$ are transformation matrices of housing ⓘ and ball ⓙ, respectively. Because of the imperfect shapes of the housing and ball, points P_i and P_j do not coincide, so there exists a tolerance between them. At point P_i, the x_i''-y_i''-z_i'' frame is defined with respect to the x_i'-y_i'-z_i' frame, whose transformation matrix is represented by $\mathbf{D}_i'$. Vector $\mathbf{c}''$ defines the relative position between points P_i and P_j in the x_i''-y_i''-z_i'' frame. Thus, vector $\mathbf{c}''$ expresses the tolerances of a spherical joint due to the imperfect shape of the housing and ball.

Constraint equations of a spherical joint with tolerances can be expressed as

$$\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j' - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i' - \mathbf{A}_i\mathbf{D}_i'\mathbf{c}_i'' = 0 \tag{15}$$

Since the time-constant vector $\mathbf{c}_i''$ represents the tolerances of the spherical joint, it can be regarded as statistically distributed random variables. Since position tolerance can occur in any direction, the three components of $\mathbf{c}_i'' = [c_{ix}'', c_{iy}'', c_{iz}'']^T$ are all independent variables, and each component can have its own tolerance $T_{c_{ix}''}$, $T_{c_{iy}''}$, $T_{c_{iz}''}$, respectively. If each element of $\mathbf{c}_i''$ follows a normal distribution and each tolerance is to be within the $\pm 3\boldsymbol{\sigma}$ range, then the mean value for each component of $\mathbf{c}_i''$ is zero and its variance can be given by Eq. (3).

A tolerance model for a revolute joint is derived. Without tolerances, a revolute joint requires five constraints equations, where three equations come from vector closure equations, and two equations define the direction of the rotational motion axis. These five constraint equations should be modified to account for the tolerances of the revolute joint. As shown in Fig. 4, the revolute joint has clearance between the inner rod and outer pipe, and the guiding surfaces between the two bodies are not uniform. Thus, there are clearances between the inner and outer bodies, and also clearance angles between two surfaces, as shown in Fig. 5. X-Y-Z is the inertial frame, x_i'-y_i'-z_i' is the local coordinates of outer pipe ⓘ, and x_j'-y_j'-z_j' is the local coordinates of inner rod ⓙ, whose transformation matrices are $\mathbf{A}_i$ and $\mathbf{A}_j$, respectively. Vectors $\mathbf{r}_i$ and $\mathbf{r}_j$ are position vectors that locate the origins of x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames with respect to the inertial frame, respectively.

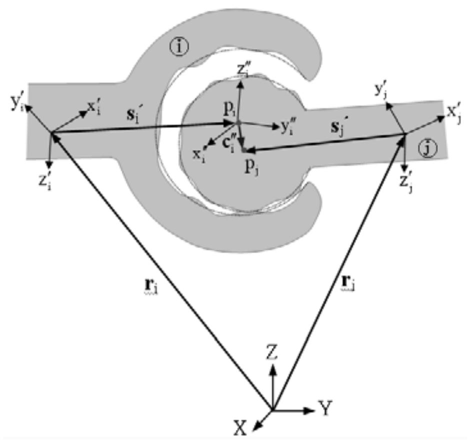


Fig. 3. Tolerance model of a spherical joint.

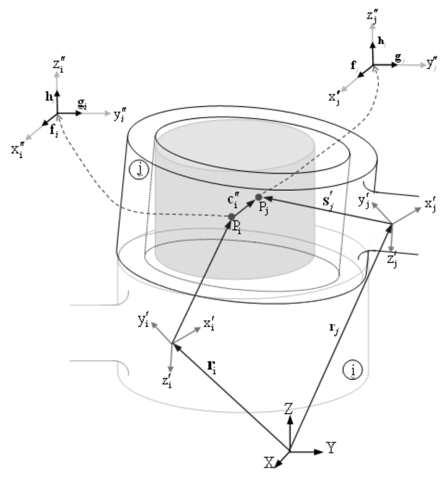


Fig. 4. Tolerance model of a revolute joint.

Points P_i and P_j are the centers of the perfectly shaped outer pipe and inner rod without any geometric defect. Points P_i and P_j would be coincident if the revolute joint were geometrically perfect. However, due to the clearance between the inner rod and outer pipe and also irregular contact surfaces, points P_i and P_j do not coincide. Local frame x_i''-y_i''-z_i'' is defined at point P_i, whose transformation matrix is represented by $\mathbf{D}_i'$, which is defined with respect to the x_i'-y_i'-z_i' frame. Similarly, at point P_j, the local frame x_j''-y_j''-z_j'' with the transformation matrix $\mathbf{D}_j'$ is defined with respect to the x_j'-y_j'-z_j' frame . Vectors $\mathbf{s}_i''$ and $\mathbf{s}_j'$ represent the locations of points P_i and P_j in x_i'-y_i'-z_i' and x_j'-y_j'-z_j' frames, respectively.

The distance between points P_i and P_j can be represented by the time-constant clearance vector $\mathbf{c}_i''$ defined in x_i''-y_i''-z_i'' coordinates. A revolute joint must satisfy the following vector closure constraint equations, which are identical to Eq. (15)

$$\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j' - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i' - \mathbf{A}_i\mathbf{D}_i'\mathbf{c}_i'' = 0 \tag{16}$$

Next, let's consider the changes in the rotational motion axis of a revolute joint due to tolerances. Vectors $\mathbf{f}_i$, $\mathbf{g}_i$, and $\mathbf{h}_i$ are the unit vectors along x_i'',y_i'',z_i'' axes, respectively, and similarly, $\mathbf{f}_j$, $\mathbf{g}_j$, $\mathbf{h}_j$ are the unit vectors along x_j'',y_j'',z_j'' axes, respectively. For a perfect revolute joint, $\mathbf{h}_i$ and $\mathbf{h}_j$ are always parallel, or equivalently, $\mathbf{g}_i^T\mathbf{h}_j = 0$ and $\mathbf{f}_i^T\mathbf{h}_j = 0$. However, due to geometric imperfection, these conditions are no more valid. As shown in Fig. 5, which shows one cross-section of the joint plane, $\mathbf{g}_i$ and $\mathbf{h}_j$ are no longer perpendicular due to product between $\mathbf{g}_i$ and $\mathbf{h}_j$ is taken, it is not zero the tolerance angle ϕ_g between $\mathbf{g}_i$ and $\mathbf{g}_j$. If the dot because $\mathbf{g}_i$ and $\mathbf{h}_j$ are no longer orthogonal.

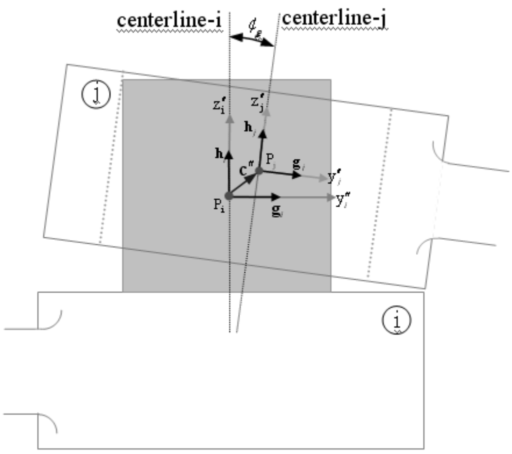


Fig. 5. Section of revolute joint.

$$\mathbf{g}_i^T\mathbf{h}_j = 1 \cdot 1\cos(90^o - \phi_g) \tag{17}$$

Similarly, a tolerance angle ϕ_f between $\mathbf{f}_i$ and $\mathbf{h}_j$ exists, and thus, the dot product between $\mathbf{f}_i$ and $\mathbf{h}_j$ is computed as

$$\mathbf{f}_i^T\mathbf{h}_j = 1 \cdot 1\cos(90^o - \phi_f) \tag{18}$$

Simplifying Eq. (17) and (18), and assembling Eq. (16)-(18), the constraint equations for a revolute joint with tolerance can be written as

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j' - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i' - \mathbf{A}_i\mathbf{D}_i'\mathbf{c}_i'' \\ \mathbf{g}_i^T\mathbf{h}_j + \sin(\phi_g) \\ \mathbf{f}_i^T\mathbf{h}_j + \sin(\phi_f) \end{bmatrix} = 0 \tag{19}$$

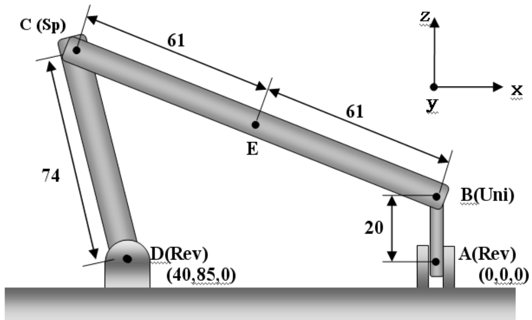
Random variables of a revolute joint are the clearance vector $\mathbf{c}_i''$ and clearance angles ϕ_f and ϕ_g. They are all independent variables with 0 mean values and their variance and tolerance relations can be similarly derived to those of a spherical joint.

4. Tolerance analysis of 4-bar mechanism

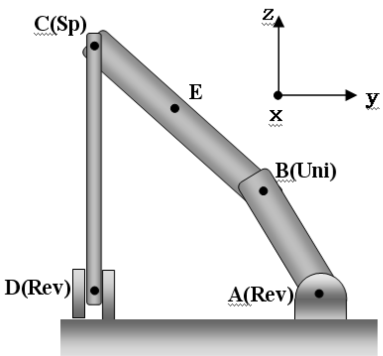
In order to verify the validity of the proposed tolerance analysis, the tolerance analysis of a spatial 4-bar mechanism, as seen in Fig. 6, is performed. The tolerances of each joint clearance and body length are defined in Table 2. If each random variable follows the normal distribution and the tolerance is to be within $\boldsymbol{\mu} \pm 3\boldsymbol{\sigma}$, the variance of each random variable can be obtained by using Eq. (3). Using the variances, the tolerances of the x, y, z components of the position at point E are computed, and the results are compared with those obtained through Monte-Carlo method [16]. The Monte-Carlo method randomly generates the design variables in Table 2 according to the normal distribution, and for each generated variable, an actual kinematic analysis is performed to compute the tolerance of the position at point E. Since the accuracy of the Monte-Carlo method depends on the number of analyses, a sufficient number of design variables should be generated.

Table 2. Data for a spatial four-bar mechanism.

	Random variables		Nominal dimension	Tolerances
	Joint & body	Directions (unit)		
1	Tolerance of spherical joint (B)	Radial (mm)	0	0.3
2	Tolerance of universal joint (C)	Radial (mm)	0	0.3
3	Clearance of spherical joint (D)	Radial (mm)	0	0.3
4		Axial (mm)	0	0.3
5		Conical (°)	0	1.72
6	Body AB	Length AB(mm)	20.0	0.3



(a) Front view



(b) Side view

Fig. 6. Spatial four-bar mechanism.

$$\mathbf{Y} = \begin{bmatrix} \mathrm{Y}_1 \\ \mathrm{Y}_2 \\ \mathrm{Y}_3 \end{bmatrix} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}_E \tag{20}$$

The procedure for the tolerance analysis is as follows. First, the constraint equations for a 4-bar mechanism are assembled, including all the design variables in Table 2. Then, $\partial\Phi/\partial q$ and $\partial\Phi/\partial b$ are computed for generalized coordinates and design variables to form Eq. (10), which can be solved for sensitivity of generalized coordinates with respect to design variables dq_p/db_i. Once dq_p/db_i are obtained; then $d\mathbf{Y}(\mathbf{q})/db_i$ can be calculated by Eq. (8) if $\partial\mathbf{Y}/\partial q_p$ and $\partial\mathbf{Y}/\partial b_i$ are known. In this example, $\partial\mathbf{Y}/\partial b_i = \mathbf{0}$ for all i since **Y,** which is the position at point E, is not directly related to design variables, and $\partial\mathbf{Y}/\partial\mathbf{q}$ is

$$\frac{\partial\mathbf{Y}}{\partial\mathbf{q}} = \begin{bmatrix} \dfrac{\partial Y_1}{\partial\mathbf{q}} \\ \dfrac{\partial Y_2}{\partial\mathbf{q}} \\ \dfrac{\partial Y_3}{\partial\mathbf{q}} \end{bmatrix} = \begin{bmatrix} 0,0,\ldots,1,0,0,\ \ldots,0 \\ 0,0,\ldots,0,1,0,\ \ldots,0 \\ 0,0,\ldots,0,0,1,\ \ldots,0 \end{bmatrix} \tag{21}$$

where the locations of non-zero element 1 correspond to the locations of x_E, y_E, and z_E in the generalized coordinate vector **q**. Finally, the variance and tolerance of Y can be obtained by Eqs. (3) and (7).

Samples of numerical data are generated by using the Monte-Carlo method to establish a nominal distribution of random variables, which are defined as Table 2. Variances are obtained by using the position of point E computed through repeated kinematic analyses performed as many times as the number of samples, and the tolerances are obtained by Eq. (3). In this case, ten thousand samples are generated by Latin-Hypercube sampling [17], which has an advantage in accuracy over the traditional method .

The tolerances of the x coordinate of point E, which are obtained by the Monte-Carlo method, and those by this work are compared in Fig. 7. The tolerance is about 1.5mm at 0° of the link AB angle, and the tolerance increases up to about 3.2mm. The tolerance of the y coordinate of point E is about 1.2mm throughout the rotation angle of link AB, as shown in Fig. 8. The tolerance of the z coordinate of point E is shown in Fig. 9. The tolerance decreases along with rotation of link AB. As shown in Fig. 7~9, the tolerances obtained by Monte-Carlo method and by this research method show almost the same results. Since the non-linearity of the function of random variables is not so severe, the tolerances obtained by the linearized function of random variables shows good agreement with real tolerances.

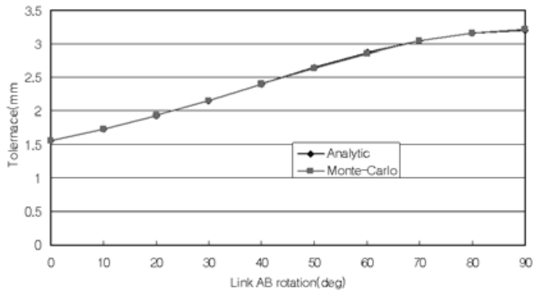


Fig. 7. Comparison of the results of mechanical errors of position Ex.

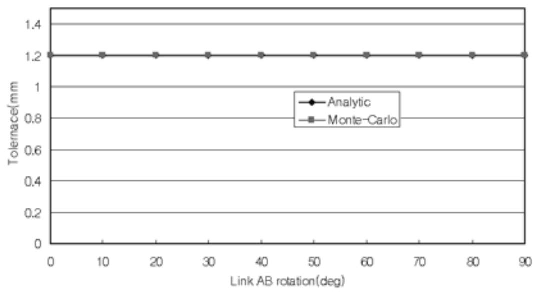


Fig. 8. Comparison of the results of mechanical errors of position Ey.

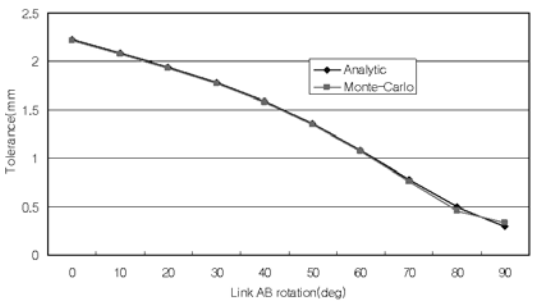


Fig. 9. Comparison of the results of mechanical errors of position Ez.

5. Optimal design of tolerance

Based on the tolerance analysis model for spatial multibody systems, an optimal design process for the tolerance control of a 4-bar mechanism is presented. The goal of tolerance optimization is to determine the design variables that can minimize the tolerances at point E and the tolerance-related cost at the same time. The optimization problem is defined as

$$\text{Min. cost}=\sum_{i=1}^{6}\frac{1}{T_i}$$

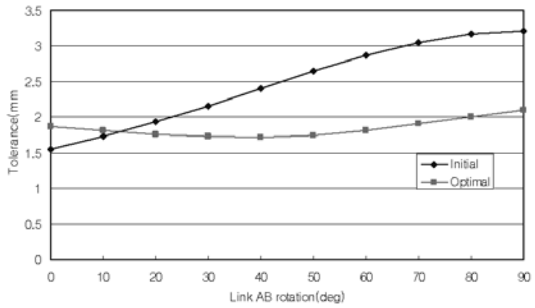


Fig. 10. Tolerance of Ex after optimization.

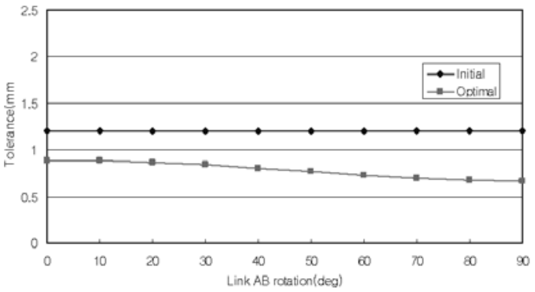


Fig. 11. Tolerance of Ey after optimization.

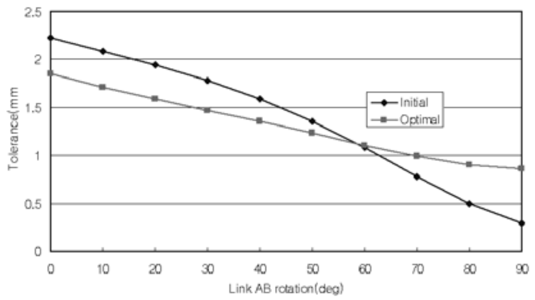


Fig. 12. Tolerance of Ez after optimization.

$$\begin{aligned} \text{subject to}\quad & 0 \le T_i \le T_{i\,upper} \qquad i=1,2,...,6 \\ & \text{Max } T_{E_x} \le 2.5 \\ & \text{Max } T_{E_y} \le 2.5 \\ & \text{Max } T_{E_z} \le 2.5 \end{aligned} \tag{22}$$

where design variable T_i represents each tolerance of Table 2. In defining the cost in Eq. (22), it is assumed that the cost for each tolerance is the inverse of the tolerance; thus, the total cost is the summation of the inverses of the tolerances. Also, the maximum tolerance at point E is limited to 2.5 mm as body AB rotates from 0° to 90°.

The tolerances of x, y, z coordinates of point E before and after optimization are shown in Figs. 10-12.

The tolerances of x, y, z coordinates of point E are decreased to 2.5mm and less. The history of the cost of production is shown in Fig. 13, which demonstrates cost reduction with iteration. The cost of tolerance and the maximum tolerance of x, y, z coordinates of point E before and after optimization are shown in Table 3. The cost of production is reduced by 49.1% and the tolerances of x, y, z coordinates of point E are decreased below 2.5mm. The tolerances before and after optimization are shown in Table 4. The shaded tolerance, which is the most sensitive to the tolerance of x, y, z coordinates of point E, is decreased, and the other less sensitive tolerances are increased.

Table 3. Cost function and tolerances of initial and optimal designs.

	Initial design	Optimal design	Reduction (%)
Cost	51.74	26.32	49.13
Maximum Tolerance of Ex	3.21(mm)	2.10	-
Maximum Tolerance of Ey	1.20(mm)	0.89	-
Maximum Tolerance of Ez	2.22(mm)	1.85	-

Table 4. Initial and optimal tolerances.

	Random variables		Tolerances	
	Joint & body	Directions (unit)	Initial	Optimal
1	Clearance of spherical joint (B)	Radial(mm)	0.30	1.37
2	Clearance of spherical joint (C)	Radial(mm)	0.30	0.45
3	Clearance of spherical joint (D)	Radial(mm)	0.30	0.81
4		Axial(mm)	0.30	0.73
5		Conical($^{\circ}$)	1.72	0.53
6	Body AB	Length AB(mm)	0.30	0.73

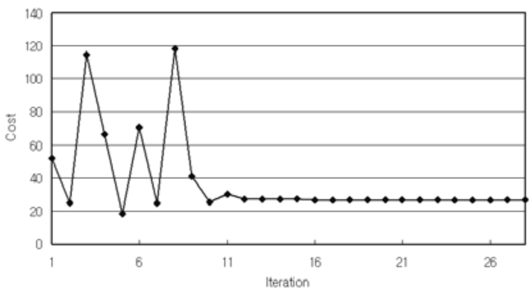


Fig. 13. History of the cost.

6. Conclusions

Taking advantage of the fact that a multibody approach is inherently suitable for tolerance analysis of mechanical systems because a multibody system can consider part level tolerance variables, a procedure for performing tolerance analysis and corresponding sensitivity analysis for spatial multibody systems was proposed. First, a formulation for multibody system tolerance analysis was developed for a detailed computational scheme to obtain total system tolerance for given part-level tolerances. In the process of computing system tolerance, the sensitivity of system tolerance with respect to part-level tolerances can be calculated with this formulation. Thus, this formulation enables the optimal design of system tolerance.

The kinematics of spatial multibody systems was redefined in terms of not only generalized coordinates but also part-level tolerance variables. Variations in the geometry of a body are specified in terms of the relative locations of joint definition points or the relative distance between them. To specify tolerances associated with various types of joints, variations in the kinematics of vector closure equations and orthogonality equations were examined, and corresponding tolerance variables were identified.

To demonstrate the validity and effectiveness of the proposed tolerance analysis procedure, tolerance analysis and tolerance optimization of a spatial 4-bar mechanism were performed. The tolerance obtained by this method was compared with the tolerance obtained by Monte-Carlo method, and they showed good agreement. Also the optimization of part tolerances to minimize the cost associated with maintaining system tolerance within an allowable range was performed.

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